

9.1 (continued)

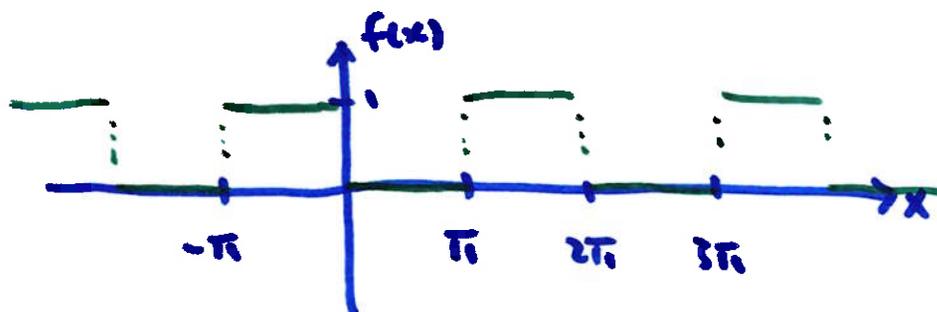
if $f(x)$ is periodic w/ period of 2π defined on $-\pi < x < \pi$,
then its Fourier series representation is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$\text{where } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n=0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad n=1, 2, 3, \dots$$

last time: $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } 0 < x < \pi \end{cases}$ period 2π



we had: $a_0 = 1$, $a_n = 0$ $n \geq 1$

$$b_n = -\frac{1}{n\pi} (1 - (-1)^n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

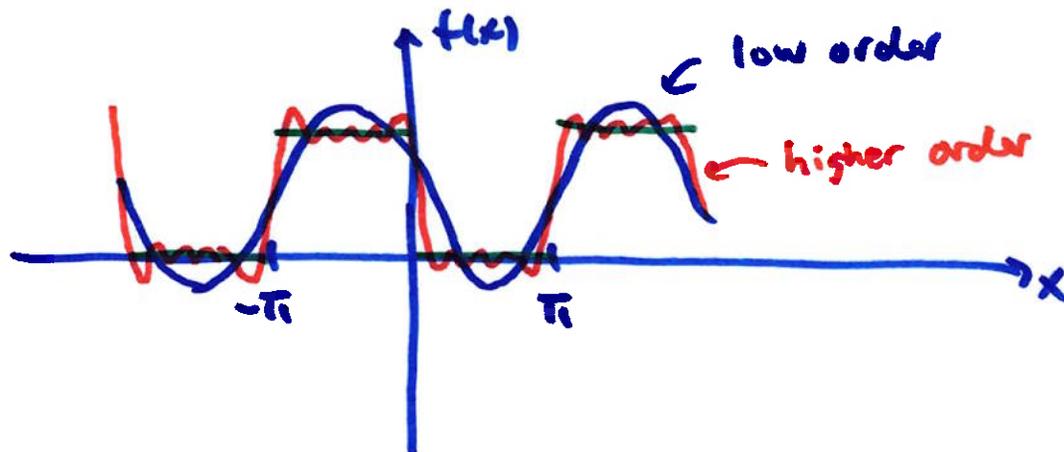
so, its Fourier series representation is

$$f(x) \sim \frac{1}{2} (1) + \sum_{n=1}^{\infty} -\frac{1}{n\pi} (1 - (-1)^n) \sin(nx)$$

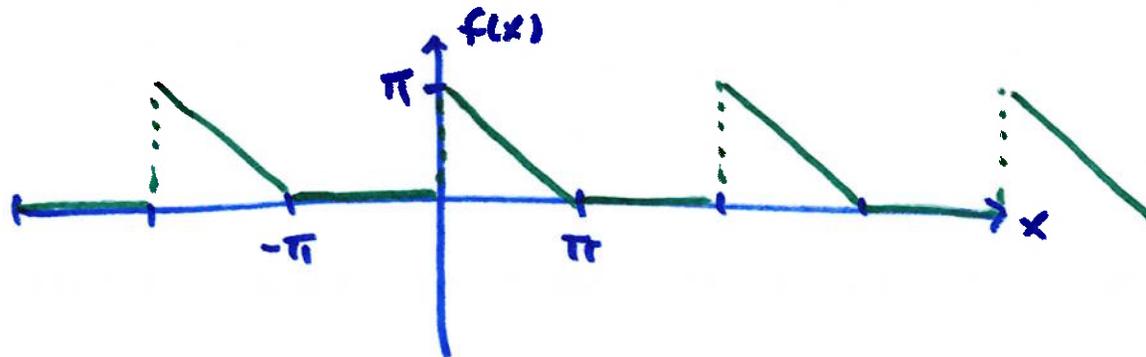
↑ converges to true $f(x)$

$$\sim \frac{1}{2} - \frac{2}{\pi} \sin(x) - \frac{2}{3\pi} \sin(3x) - \frac{2}{5\pi} \sin(5x) - \dots$$

just like Taylor series, the more terms we include the better the Fourier series resembles the true $f(x)$



let's look at $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$ period 2π



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

calculate a_0 separately

$$a_0 = \frac{1}{\pi} \underbrace{\int_{-\pi}^{\pi} f(x) dx}_{\text{area under } f(x)} = \frac{1}{\pi} \left(\frac{1}{2} \cdot \pi \cdot \pi \right) = \frac{1}{2} \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad \text{by parts}$$

= ...

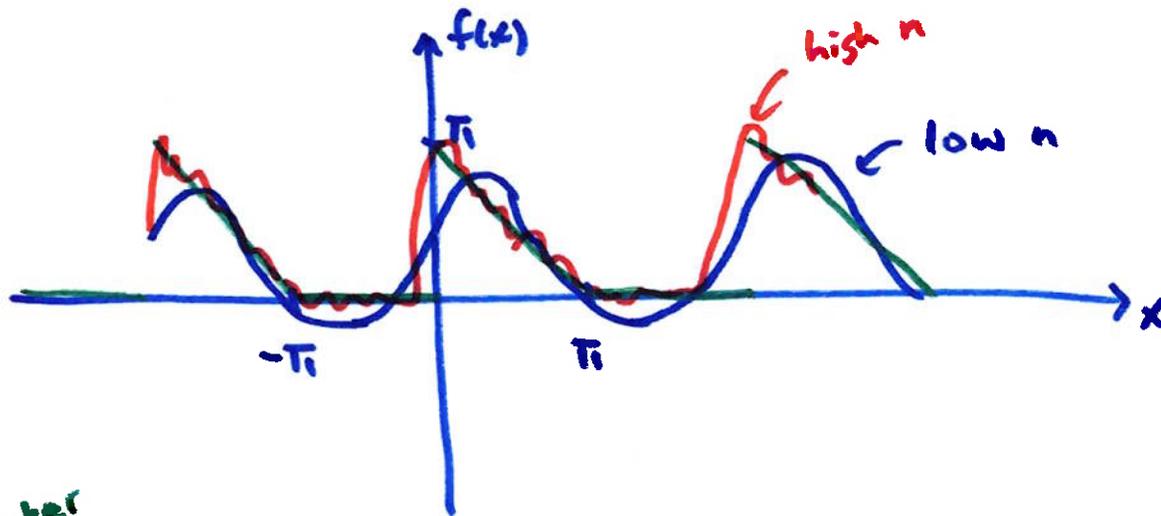
$$= \frac{1}{n^2 \pi} (1 - (-1)^n) \quad n=1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad \text{by parts}$$

$$= \dots = \frac{1}{n} \quad n=1, 2, 3, \dots$$

$$f(x) \sim \frac{1}{2} \left(\frac{1}{2} \pi \right) + \sum_{n=1}^{\infty} \left[\frac{1}{n^2 \pi} (1 - (-1)^n) \cos(nx) + \frac{1}{n} \sin(nx) \right]$$

$$\sim \frac{\pi}{4} + \underbrace{\frac{2}{\pi} \cos(x) + \sin(x)}_{n=1} + \underbrace{\frac{1}{2} \sin(2x)}_{n=2} + \underbrace{\frac{2}{9\pi} \cos(3x) + \frac{1}{3} \sin(3x)}_{n=3} + \dots$$

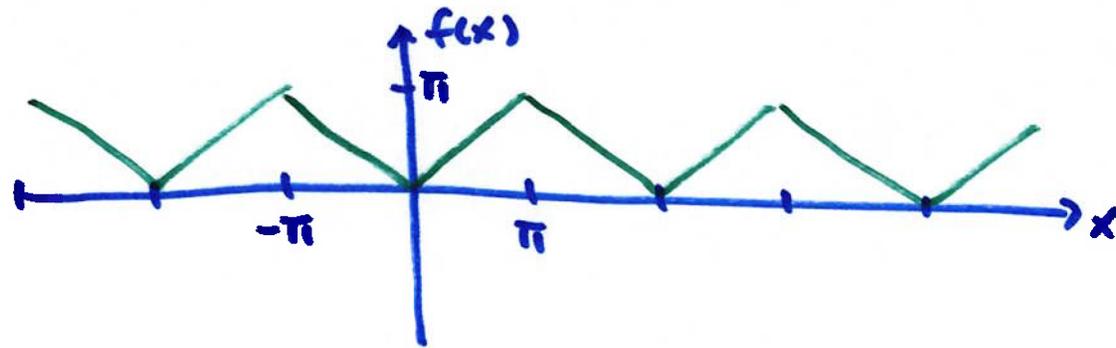


Fourier series
properties
discussed later

notice the overshoot right before and after a discontinuity that does not go away even with high n

also, Fourier series seems cut through the middle at discontinuity

try this one: $f(x) = |x|$ $-\pi < x < \pi$ period 2π



$$f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left(\frac{1}{2} \cdot 2\pi \cdot \pi \right) = \pi$$

$$a_n = \dots = \frac{-2}{n\pi} (1 - (-1)^n)$$

$$b_n = \dots = 0$$